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A Solution to the Inverse Problem of Mathematical Modeling of Structural Mechanics for Beam Elements

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Abstract. The solution of the direct problem of mathematical modeling (DPMM) is a well-studied subject – at present there is a wide variety of software packages aimed at solving it in one degree or another. However, of great practical interest is the inverse problem of mathematical modeling (IPMM), whose solution presents considerably more difficulties than DPMM. Due to the complexity of the IPMM problem, there is currently no software product that can effectively obtain its solution. This study concerns the problem of solving the inverse problem of mathematical modeling in the context of statically indeterminate mechanical systems modeled by finite beam elements.

As a consequence of the development of contemporary industry and construction taking place against the background of a constant lack of resources, the problem of optimal design is one of the most pressing challenges facing humanity. In the context of mechanical system design, one of the most important criteria is the cost of the final product. This criterion becomes especially important in the design of buildings, since buildings are mechanical systems of colossal dimensions, consisting of individual elements and structures whose nonoptimal design could lead to a significant reduction of the attractiveness of the building in terms of investment.

Unfortunately, despite considerable advances in the area of mathematical programming, the task of finding the optimal configuration remains a complicated and multifaceted problem. The search for the optimal configuration involves the inverse problem of mathematical modeling (IPMM), the solution of which is several orders of magnitude more complex than the solution of the direct problem of mathematical modeling. The main difficulties of IPM consist for the main part in optimal solutions and non-linearity of the equations obtained; the absence of fast and flexible way of solving the general nonlinear programming problem (finding the global optimum when the objective function is not convex). For the end user-designer, this is evinced in the almost complete absence of software solving IPMM in the context of mechanical systems, whereas software systems solving the direct problem of mathematical modeling are already quite common. In the overwhelming majority of cases, these software packages use the finite element method for solving MMDP.

In this paper, we will consider the application of nonlinear mathematical optimization to the solution of the inverse problem of mathematical modeling in structural mechanics on the example of the determination of the sectional parameters for circular cross-section beam elements. In order to determine the extent of the rationality of the cross-section, it is necessary to enter certain criteria, whose extreme value we will attempt to find.

One of the most significant and obvious criteria is the minimization of the total mass of the system. This problem is formulated in relation to a frame structure.

Let there be a set of structural elements Θ with a thickness N. Each set element is characterized by the following parameters:

l_i – element length;
A_i – element area;
ρ_i – density of the element material;
E_i – Young's modulus of the element;
$[\sigma]_i$ – ultimate strength under normal element stresses

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$[\tau]_i$ – ultimate tensile strength under shear element stresses.

Then, the total mass of the system can be defined as

$$M = \sum_{i=1}^{N} A_i l_i \rho_i.$$

The most advantageous configuration is the configuration that minimizes the total mass, in other words,

$$M \rightarrow min.$$

However, it is necessary to take into account the fact that any system under consideration in the context of structural mechanics is subject to restrictions, significantly reducing the set of feasible solutions. The restrictions imposed on each element of the system express some requirements that it must meet, namely,

– satisfaction of the strength conditions characteristic of this material. Civil engineering practice is typically limited by the condition of normal and shear stresses, defined by the following dependencies:

$$|\sigma|_{max} \leq [\sigma], \quad |\tau|_{max} \leq [\tau],$$

where

$|\sigma|_{max}$ is maximum normal stress;

$|\tau|_{max}$ is maximum shear stress;

satisfaction of the displacement constraint condition;

satisfaction of the stability condition.

Subsequently, we will confine ourselves to a consideration of beam elements with the condition of ensuring strength.

In view of the above, we can formulate the problem of optimization of the frame structure configuration.

$$M = \sum_{i=1}^{N} A_i l_i \rho_i \rightarrow min,$$

$$|\sigma_i|_{max} \leq [\sigma]_i, \qquad |\tau_i|_{max} \leq [\tau]_i.$$

Our goal will be to find the optimal and acceptable system configuration. In terms of subtasks, the need arises to determine the internal forces applied to the elements of the system, on the basis of which a satisfactory level of connectors can be determined. In order to solve this subtask, it is possible to use, in particular, the finite element method (FEM), as well described in [2]. In this case it will be necessary to solve the subtask of the following type:

$$K\bar{u} - \bar{B} = 0,$$

where K is the global stiffness matrix of the system; $\bar{B}$ is the force vector; $\bar{u}$ is the generalized system parameter vector.

If in terms of generalized parameters u deflections and angles of rotation of the section at the ends of the element are chosen, then this parameter vector is expressed as follows: $\tilde{u} = [\, u_i, \theta_i, u_{i+1}, \theta_{i+1}]$, and the corresponding basis functions

$$\varphi_1 = 1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}, \;\; \varphi_2 = \left(\frac{x}{l} - 2\frac{x^2}{l^2} + \frac{x^3}{l^3}\right) l, \;\; \varphi_3 = 3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}, \;\; \varphi_4 = \left(\frac{x^3}{l^3} - \frac{x^2}{l^2}\right) l,$$

where l is element length.

For a linear beam element, the maximum normal stresses can be identified as

$$\sigma_{max} = \frac{M_{max}}{I} y_{max},$$

here M_{max} is the maximum bending moment in the body of the rod element;

y_{max} is the distance from the neutral section line to the most distal point of the section.

In this case, since

$$M = EI\, u_{xx},$$

we have

$$\sigma_{max} = \frac{M_{max}}{I} y_{max} = \frac{I\,|u_{xx}|_{max}}{I} E y_{max} = |u_{xx}|_{max}\, E y_{max}.$$

Maximum shear stresses can be found using the formula

$$\tau_{max} = \frac{Q_{max}\, S_y}{I\, b},$$

where

Q_{max} is the maximum transverse force in the body of the rod member;

S_y is the static moment of the vulnerable half-section of the rod;

b is the minimum width at the boundary of the vulnerable half-section of the rod.

In this case, since

$$Q = EI\, u_{xxx},$$

we have

$$\tau_{max} = \frac{Q_{max}\, S_y}{I\, b} = \frac{I\,|u_{xxx}|_{max}}{I} E \frac{S_y}{b} = |u_{xxx}|_{max}\, E \frac{S_y}{b}.$$

Then the conditions for guaranteeing strength can be presented in the following form:

$$|u_{xx,max}|\, E y_{max} \leq [\sigma];$$
$$\left|u_{xxx,max}\right| E \frac{S_y}{b} \leq [\tau].$$

If we introduce the vector

$$\bar{c}_\sigma = \left[\frac{12\,x}{l^3} - \frac{6}{l^2}, \frac{6x}{l^2} - \frac{4}{l}, \frac{6}{l^2} - \frac{12x}{l^3}, \frac{6x}{l^2} - \frac{2}{l}\right]^T, x \in [0 \ldots l];$$
$$\bar{c}_\tau = \left[\frac{12}{l^3}, \frac{6}{l^2}, -\frac{12}{l^3}, \frac{6}{l^2}\right]^T,$$

these conditions may be converted to the form

$$|(\bar{c}_\sigma, \tilde{u})| \cdot y_{max} \leq \frac{[\sigma]}{E};$$
$$|(\bar{c}_\tau, \tilde{u})| \cdot \frac{S_y}{b} \leq \frac{[\tau]}{E},$$

or

$$(\bar{c}_\sigma(x), \tilde{u}) \cdot y_{max} \pm \frac{[\sigma]}{E} \leq 0, \quad x \in [0 \ldots l];$$
$$(\bar{c}_\tau, \tilde{u}) \cdot \frac{S_y}{b} \pm \frac{[\tau]}{E} \leq 0.$$

The vector $\bar{c}_\sigma$ is a linear vector function of the parameter x, and it can reach a maximum only at the ends; therefore, we can substitute this vector by two vectors:

$$\bar{c}_\sigma^1 = \bar{c}_\sigma(0) = \left[-\frac{6}{l^2}, -\frac{4}{l}, \frac{6}{l^2}, -\frac{2}{l}\right];$$
$$\bar{c}_\sigma^2 = \bar{c}_\sigma(l) = \left[\frac{6}{l^2}, \frac{2}{l}, -\frac{6}{l^2}, \frac{4}{l}\right].$$

Thus our optimization problem obtains the following form:

$$M = \sum_{i=1}^{N} A_i l_i \rho_i \rightarrow min;$$
$$\left(\bar{c}_{\sigma,i}^1, \tilde{u}_i\right) \cdot y_{max,i} \pm \frac{[\sigma]_i}{E_i} \leq 0;$$

$$\left(\bar{c}^{2}_{\sigma,i_i}, \tilde{u}_i\right) \cdot y_{max,i} \pm \frac{[\sigma]_i}{E_i} \le 0;$$
$$\left(\bar{c}_{\tau,i}, \tilde{u}_i\right) \cdot \frac{S_{y,i}}{b_i} \pm \frac{[\tau]_i}{E_i} \le 0.$$

Now, consider the case when each element consists of a round rod. In this case,

$$A_i = \frac{\pi}{4} d_i^2;\ I_i = \frac{\pi}{64} d_i^4;\ y_{max,i} = \frac{d_i}{2};\ S_i = \frac{d^3}{12}; b_i = d.$$

Then our problem takes the following form:

$$M = \sum_{i=1}^{N} A_i l_i \rho_i = \sum_{i=1}^{N} \frac{\pi}{4} d_i^2 l_i\, \rho_i \to min;$$
$$\Psi_i = \begin{bmatrix} \left(\bar{c}^{1}_{\sigma,i}, \tilde{u}_i\right) \cdot \frac{d_i}{2} \pm \frac{[\sigma]_i}{E_i} \\ \left(\bar{c}^{2}_{\sigma,i_i}, \tilde{u}_i\right) \cdot \frac{d_i}{2} \pm \frac{[\sigma]_i}{E_i} \\ \left(\bar{c}_{\tau,i}, \tilde{u}_i\right) \cdot \frac{d_i^2}{12} \pm \frac{[\tau]_i}{E_i} \end{bmatrix} \le 0$$

The function M can be represented as

$$\psi(\bar{d}) = \frac{1}{2}\left(Q\bar{d}, \bar{d}\right),$$

where

$$Q_{i,j} = \begin{cases} \frac{\pi}{2} \rho_i l_i, & i = j \\ 0, & i \ne j \end{cases}.$$

As can be seen, the function ψ is quadratic and convex, with an absolute and single minimum at point

$$\bar{d} = \emptyset.$$

Thus, we can formulate our optimization problem in canonical form as

$$\frac{1}{2}\left(Q\bar{d}, \bar{d}\right) \to \min,$$
$$\bar{\Xi} = K(\bar{d})\bar{u} - \bar{B} = 0,$$
$$\bar{\Psi}(\bar{u}, \bar{d}) \le 0,$$
$$-\bar{d} \le 0.$$

This problem can be solved using the sequential quadratic programming (SQP) method [3]. The SQP method is one of the most common methods for obtaining solutions of nonlinear programming problems. One of the most efficient implementations in the case of nonlinear programming tasks having large dimensions is presented in [4]. In order to apply SQP to our problem, it is necessary to linearize the connections

$$\bar{\Psi}(\bar{d}) \approx \bar{\Psi}(\bar{d}_0) + \bar{\Psi}_{\bar{d}}(\bar{d}_0)\bar{d},$$
$$\bar{\Xi}(\bar{d}) \approx \bar{\Xi}(\bar{d}_0) + \bar{\Xi}_{\bar{d}}(\bar{d}_0)\bar{d}.$$

In this case, our task becomes a quadratic programming problem, which can be solved using the conditional gradient or the Dantzig-Wolfe decomposition method. These methods allow the quadratic programming problem to be reduced to a linear programming problem.

In the future it is planned to develop a program that implements the scheme described above. In the solution it is intended to use such properties of the problem as the sparsification of the obtained matrices being investigated, as well as the fact that it reduces well to the task of linear programming, making it possible to utilize effectively the possibilities of contemporary heterogeneous systems. There are a number of studies on the possibility of using heterogeneous systems for solving linear programming problems; in particular, [1] and [5] presented algorithms for solving linear programming problems on GPGPU systems from NVIDIA with CUDA (Compute Unified Device

Architecture) API. It is planned to transfer these algorithms on OpenCL and apply them to the solution of the problem under investigation.

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